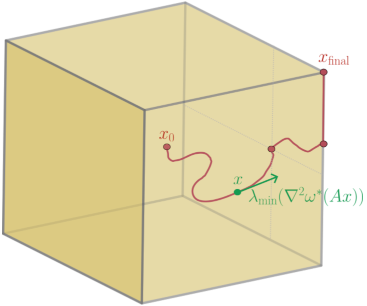
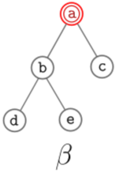
Up to the constant: discrepancy theory and iterative algorithms on random matrices



$$Z_a^\alpha = \sum_{\substack{b: \\ a,b \text{ distinct}}} A_{ab}$$

$$Z_a^\beta = \sum_{\substack{b,c,d,e: \\ a,b,c,d,e \text{ distinct}}} A_{ab}A_{bd}A_{be}A_{ac}$$



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Discrepancy Theory (1/2)

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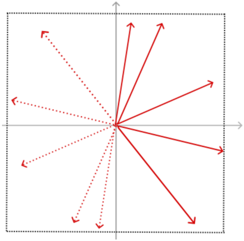

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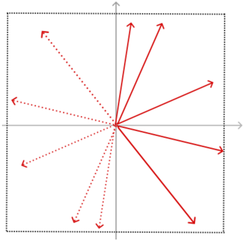
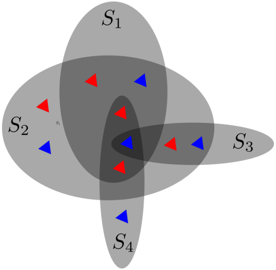


Ex: set system $S_1, \ldots, S_d$ over n elements

$$u_{i,j} = \begin{cases} 1 & \text{if } i \in S_j \\ 0 & \text{otherwise} \end{cases}$$

Goal: red/blue coloring with small maximal imbalance



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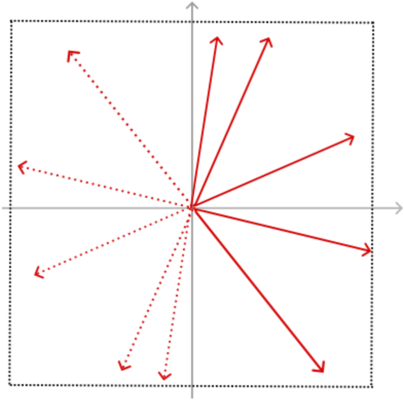

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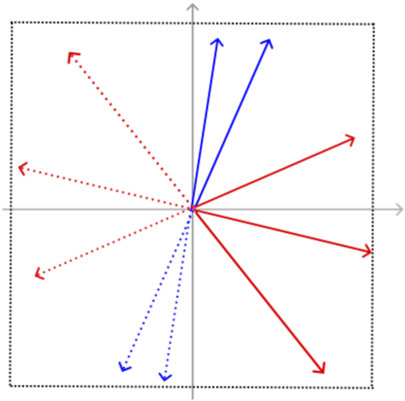

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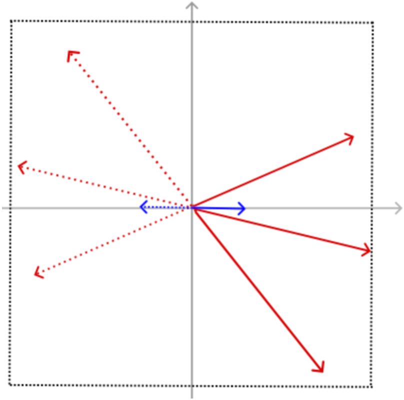

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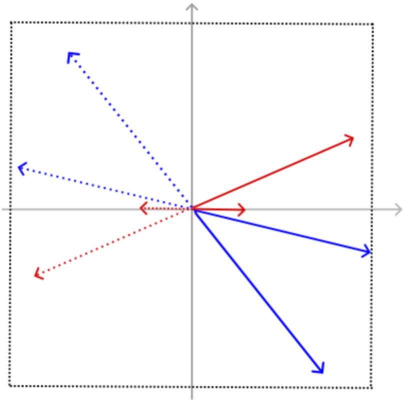

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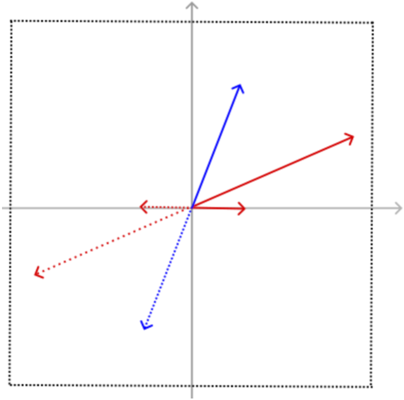

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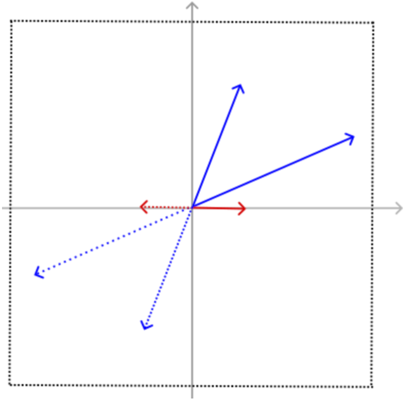

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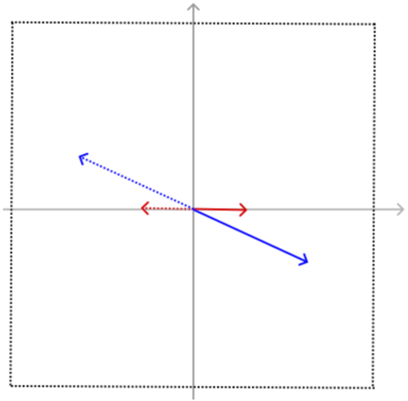

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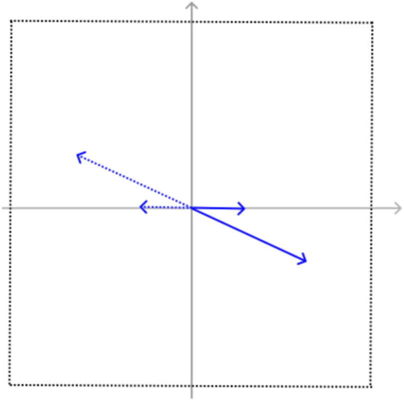

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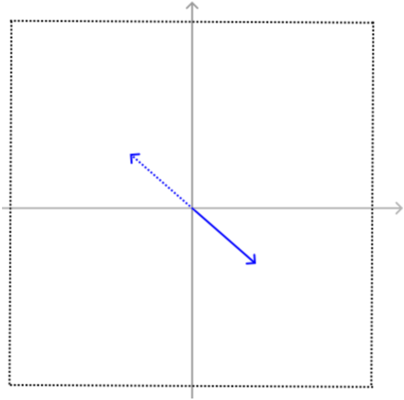

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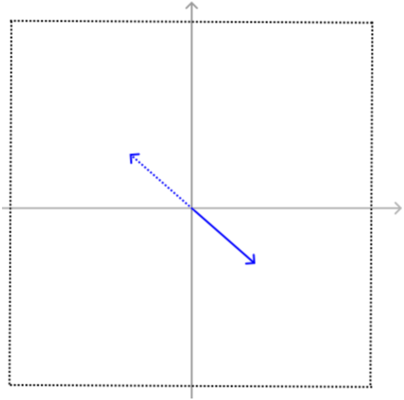

Spencer's Theorem

High-dimensional regime: $d = n$

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Today: algorithmic proof of this theorem improving the 6

Sparsification via Discrepancy

Discrepancy: given $\mathcal{O}_1, \ldots, \mathcal{O}_n$, find $x_1, \ldots, x_n \in \{-1, 1\}$ s.t. $\left\| \sum_i x_i \mathcal{O}_i \right\|$ is small

$\longrightarrow$

Sparsification: given $\mathcal{O}_1, \ldots, \mathcal{O}_n$, find $S \subseteq [n]$ s.t. $|S| \ll n$ and $\sum_{i \in S} \mathcal{O}_i \approx \sum_{i \in [n]} \mathcal{O}_i$

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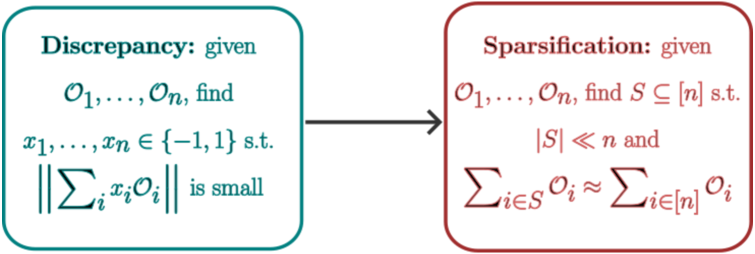
Why sparsification?

- fast algorithms
- low memory

Sparsification via Discrepancy



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1. Solve the discrepancy problem to get $\mathbf{x}$
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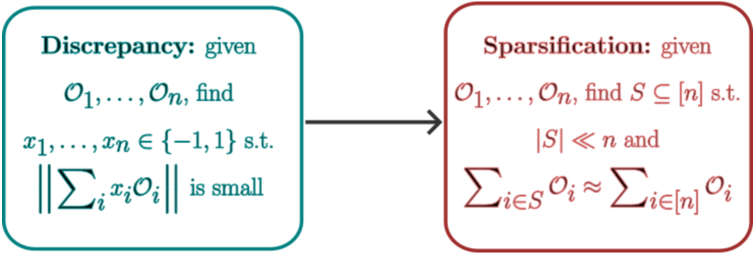
3. Repeat!

Sparsification via Discrepancy



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Examples:

- [Reis–Rothvoss'20; Jambulapati–Reis–Tian'24] sparsify graphs
- [Reis–Rothvoss'22] sparsify convex combinations
- [Bozzai–Reis–Rothvoss'23] sparsify zonotopes

Discrepancy and Continuous Methods

Thm: [Spencer'85] For any $\mathbf{u}_1, \ldots, \mathbf{u}_n$ s.t. $\|\mathbf{u}_i\|_\infty \leqslant 1$, there exists $\mathbf{x} \in \{\pm 1\}^n$ s.t. $\|\sum_i x_i \mathbf{u}_i\|_\infty = O(\sqrt{n})$

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Already different known *algorithmic* proofs:

- [Bansal'10, Lovett–Meka'12] random walks, SDP
- [Eldan–Singh'14, Rothvoss'14] LP with random objective
- [Levy–Ramadas–Rothvoss'17] multiplicative weights update
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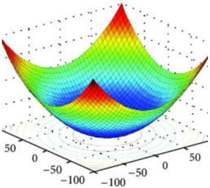
More and more inspired by continuous optimization



Today: Newton's method on a regularized objective

Spencer's Theorem via Regularization (1/3)

Thm: [Spencer'85] For any $\mathbf{A} \in \mathbb{R}^{n \times n}$ s.t. $|A_{ij}| \leqslant 1$, there exists $\mathbf{x} \in \{\pm 1\}^n$ s.t. $\|\mathbf{Ax}\|_\infty = O(\sqrt{n})$

1. Define a "smooth" proxy $\omega^*(\cdot)$ for $\|\cdot\|_\infty$
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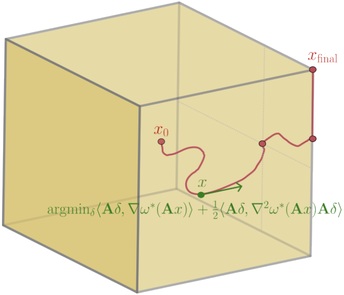

Algorithm:

Start from $\mathbf{x} = (0, \ldots, 0)$

While $\mathbf{x} \notin \{\pm 1\}^n$

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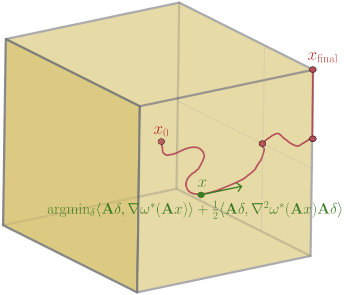

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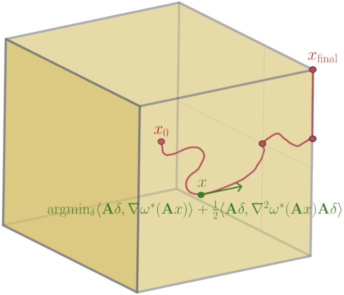

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- $\mathbf{x}$ makes a small step in direction δ while staying in $[-1, 1]^n$

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Thm: [Spencer'85] For any $\mathbf{A} \in \mathbb{R}^{n \times n}$ s.t. $|\mathbf{A}_{ij}| \leqslant 1$, there exists $\mathbf{x} \in \{\pm 1\}^n$ s.t. $\|\mathbf{A}\mathbf{x}\|_\infty = O(\sqrt{n})$

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Up to tracking $\begin{bmatrix} +\mathbf{A} & \mathbf{0} \\ -\mathbf{A} & \mathbf{0} \end{bmatrix}$, assume WLOG

$$\|\mathbf{A}\mathbf{x}\|_\infty = \max_{1 \leqslant i \leqslant n} (\mathbf{A}\mathbf{x})_i = \max_{\mathbf{r} \in \Delta_n} \langle \mathbf{A}\mathbf{x}, \mathbf{r} \rangle$$

where $\Delta_n := \{\mathbf{r} \in \mathbb{R}^n : r_i \geqslant 0, \sum_i r_i = 1\}$.

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Up to tracking $\begin{bmatrix} +\mathbf{A} & \mathbf{0} \\ -\mathbf{A} & \mathbf{0} \end{bmatrix}$, assume WLOG

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Spencer's Theorem via Regularization (2/3)

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3. Worst case: one coordinate of $\mathbf{x}$ get frozen every time $\|\mathbf{x}\|_2^2$ increases by 1, total cost
$$1 \times \frac{1}{\sqrt{n}} + 1 \times \frac{1}{\sqrt{n-1}} + \ldots = O(\sqrt{n}). \quad \square$$

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Thm: [P.–Vladu'23] For any $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^n$ s.t. $\|\mathbf{u}_i\|_\infty \leqslant 1$, there exists $x_1, \ldots, x_n \in \{-1, 1\}$ s.t. $\|\sum_i x_i \mathbf{u}_i\|_\infty \leqslant 3.8\sqrt{n}$

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[Audibert–Bubeck'09; Allen-Zhu–Liao–Orecchia'15]

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Regularized Newton's method with smax ≈ multiplicative weights update ≈ derandomizing the coin flipping argument

Matrix Spencer conjecture

Conjecture: For any symmetric $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathbb{R}^{n \times n}$ s.t. $\|\mathbf{A}_i\|_{\mathrm{op}} \leqslant 1$, there exists $x_1, \ldots, x_n \in \{-1, 1\}$ s.t. $\|\sum_i x_i \mathbf{A}_i\|_{\mathrm{op}} = O(\sqrt{n})$

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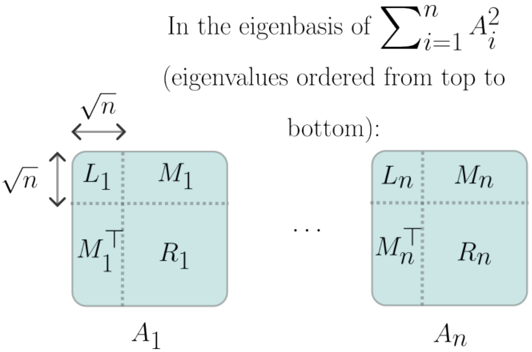

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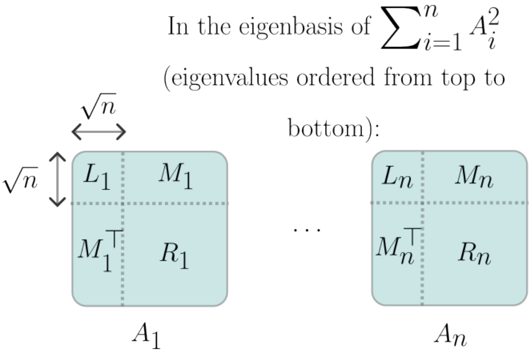

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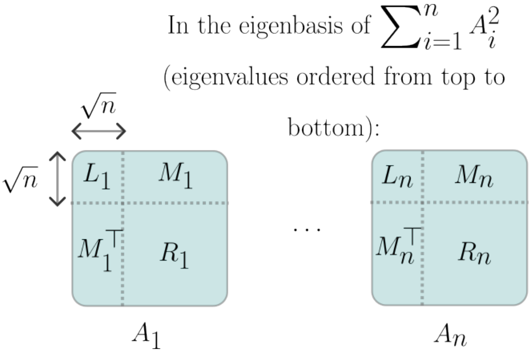

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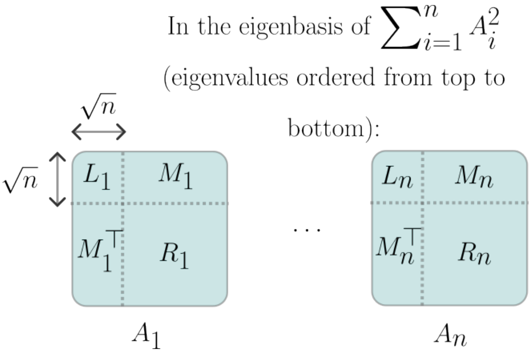

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Open question 3: same if $\mathrm{rank}(\mathbf{A}_i) \ll n$? [Bansal–Jiang–Meka'23]

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Problem: solve for $\mathbf{x}$ in

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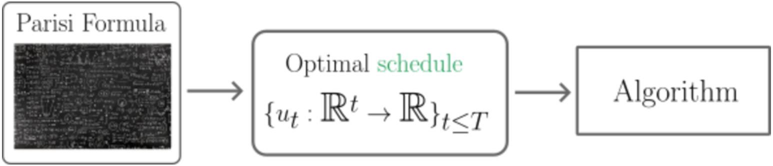

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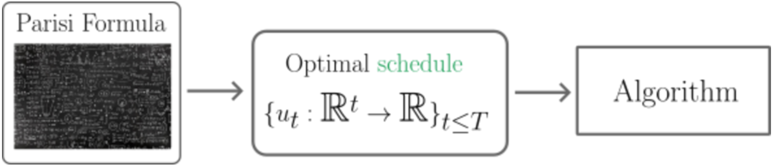

Optimal schedule: given i.i.d. $g_t \sim \mathcal{N}(0,1)$,

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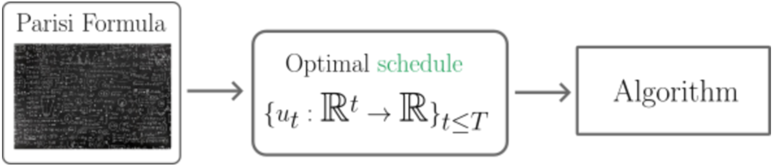

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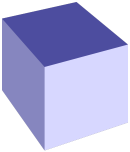

Question: How to pick the g_t's?

Diagram analysis of iterative algorithms

Perspective: “symmetrized” Fourier analysis

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α

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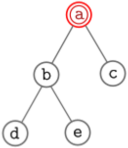

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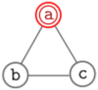

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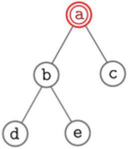

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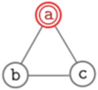

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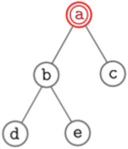

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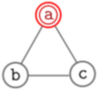

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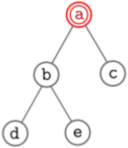

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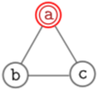

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3. As $n \to \infty$, the trees where the root has degree 1 are independent Gaussian vectors.

Asymptotic diagrams cheatsheet (1/2)

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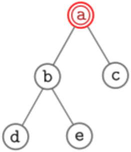
α

$$Z_a^\beta = \sum_{\substack{b,c,d,e: \\ a,b,c,d,e \text{ distinct}}} A_{ab}A_{bd}A_{be}A_{ac}$$



β

Asymptotic diagram basis $\{\mathbf{Z}^\alpha : \alpha \text{ rooted tree}\}$

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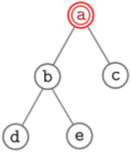

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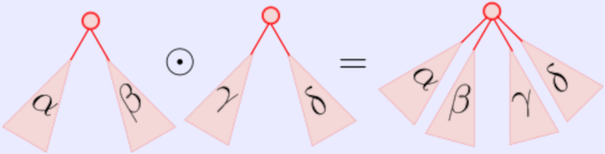
$$\text{if } \{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset:$$



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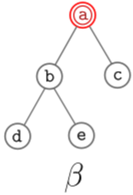
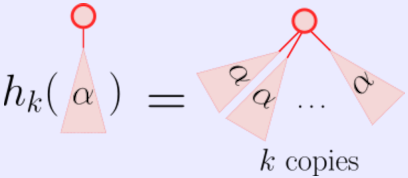


Asymptotic diagram basis $\{\mathbf{Z}^\alpha : \alpha \text{ rooted tree}\}$

Ex:



Asymptotic diagrams cheatsheet (2/2)

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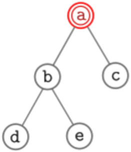

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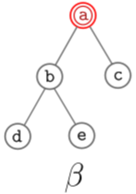

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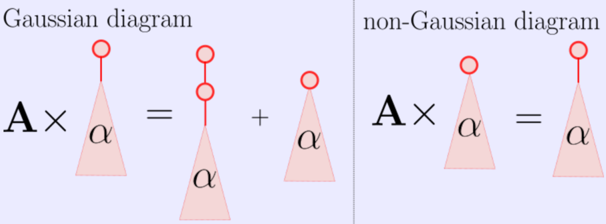

Back to optimizing random quadratic forms

Input: optimal schedule $\{u_t : \mathbb{R}^t \to \mathbb{R}\}_{t \leqslant T}$ s.t. given i.i.d. $g_t \sim \mathcal{N}(0,1)$,

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Upcoming challenges

Applies to *generalized first order methods* [Montanari–Celentano–Wu'20]

Ex: approximate message passing, power iteration, ...

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Theorem 3.35 (State evolution for GFOM). *Assume Assumption 3.1 on A and Assumption 3.34 on $f_0, f_1, \ldots, f_t$. Generate $x_t, y_t \in \mathbb{R}^n$ using the GFOM*

$$x_0 = \vec{1} \qquad y_t = Ax_t \qquad x_{t+1} = f_t(y_t, \ldots, y_0)\,.$$

Let $X_t, Y_t \in \Omega$ be the result of running the GFOM operations asymptotically using the rules in Section 2.3. Then for all polynomial functions $\psi : \mathbb{R}^{2(t+1)} \to \mathbb{R}$,

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Open question 4: how about $\Omega(\log n)$ iterations?

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Open question 1: tight constant in Spencer's theorem?

Open question 2: other useful regularizers?

Open question 3: matrix Spencer up to polylog rank using regularized Newton's method?

Open question 4: diagram analysis of iterative algorithms for $\Omega(\log n)$ iterations?