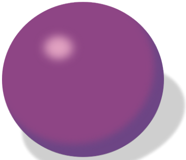
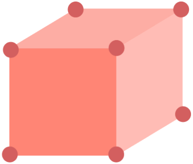
On maximizing polynomials



$$\max_{\|\mathbf{x}\|_2=1} p(\mathbf{x})$$



$$\max_{\mathbf{x}\in\{-1,1\}^n} p(\mathbf{x})$$

based on joint works with

Tim Hsieh

CMU

Chris Jones

Bocconi

Pravesh Kothari

Princeton

Lucas Pesenti

Bocconi

Luca Trevisan

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Motivation (1/2)

Maximize over x: $p(x_1, \ldots, x_n) := \sum_i c_i x_i + \sum_{i<j} c_{ij} x_i x_j + \sum_{i<j<k} c_{ijk} x_i x_j x_k$

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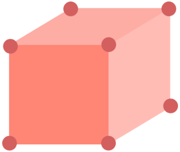

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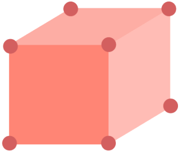

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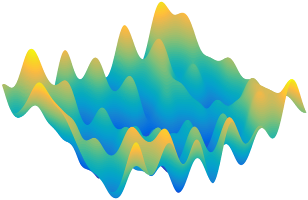

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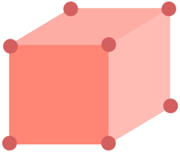

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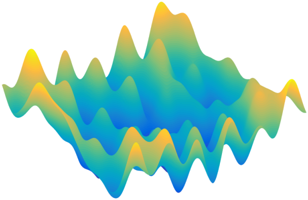

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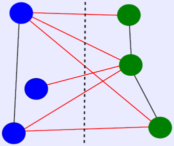
Ex: Max-Cut



Fraction of edges cut by $\mathbf{x}$:
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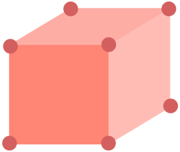

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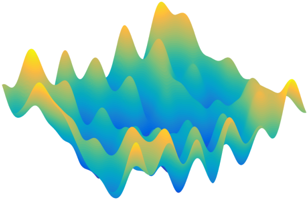

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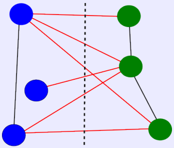

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$$p(\mathbf{x}) = \sum c_{ijk} x_i x_j x_k$$

with $c_{ijk} = x_i^* x_j^* x_k^* + \text{noise}$

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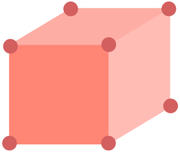

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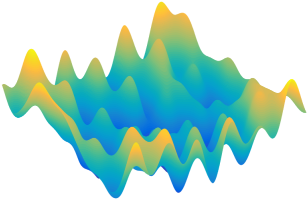

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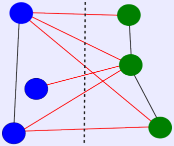

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Ex: Max-3SAT

$(x_1 \vee \neg x_3 \vee x_4)$

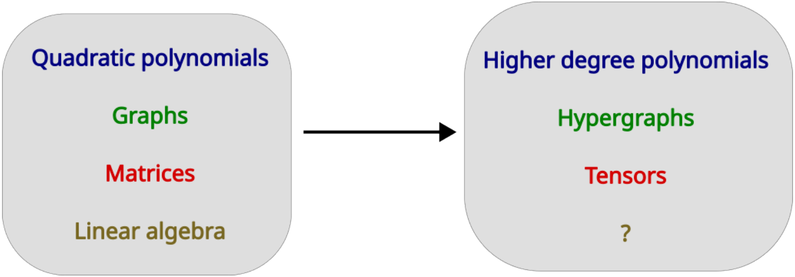
$\wedge\ (\neg x_2 \vee x_3 \vee \neg x_5)$

$\wedge\ \ldots$

Fraction of clauses satisfied by $\mathbf{x}$:

$\frac{7}{8} + p_1(\mathbf{x}) + p_2(\mathbf{x}) + p_3(\mathbf{x})$

Motivation (2/2)



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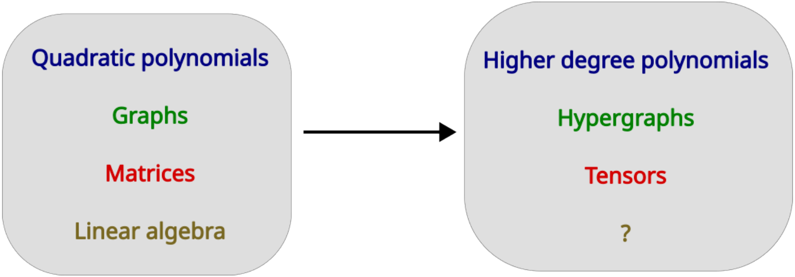
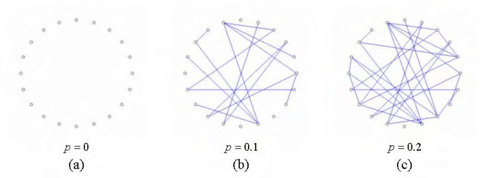


Today:

1. Spectral theory for hypergraphs
2. Approximation algorithms
3. Optimizing random polynomials

1. Spectral theory for hypergraphs

Spectral bound for random hypergraphs



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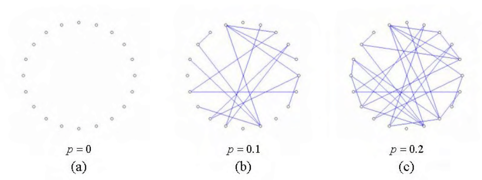
Spectral bound for random hypergraphs



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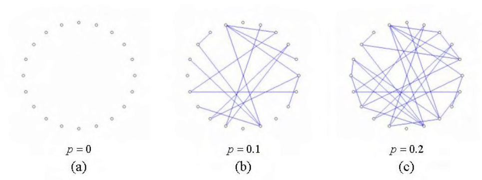

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$$\|\mathbf{A}\| \asymp \mathrm{Tr}\left(\mathbf{A}^{2k}\right)^{\frac{1}{2k}} \text{ if } k \sim \log n\,.$$

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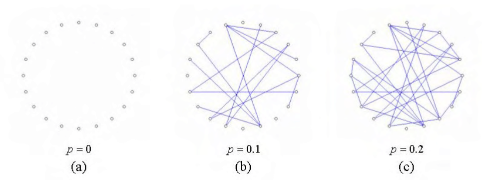

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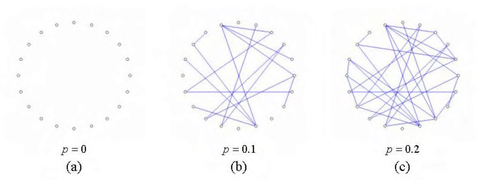

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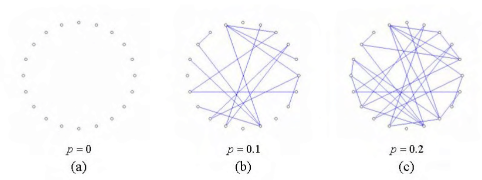

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✗ Flattenings

✓ Union bounds/Chaining

2. Approximation algorithms

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October 3, 2023

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✗ [Hastad'01] $\frac{7}{8}+\epsilon$ is NP-hard.

✓ [Hsieh-Kothari-P.-Trevisan'24] $\frac{7}{8}+\widetilde{\Omega}(\frac{1}{n^{3/4}})$

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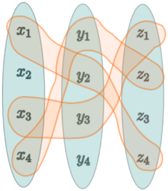

Decoupling:

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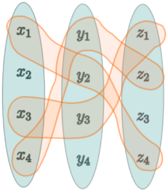

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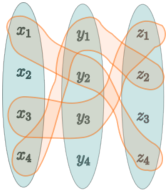

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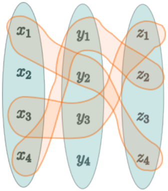

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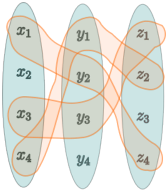

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3. Optimizing random polynomials

Optimizing random polynomials

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 Typical/random $c_{ij} \sim \{-1, 1\}$?

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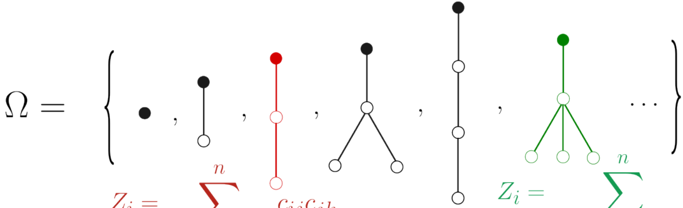
Physics: ✓ simple ✗ non-rigorous

Maths: ✓ rigorous ✗ technical

The Fourier tree basis



$$Z_i = \sum_{\substack{j,k=1 \\ i,j,k \text{ distinct}}}^{n} c_{ij}c_{jk}$$

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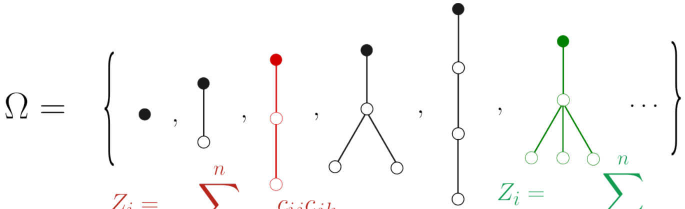

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[Jones-P.'24+]

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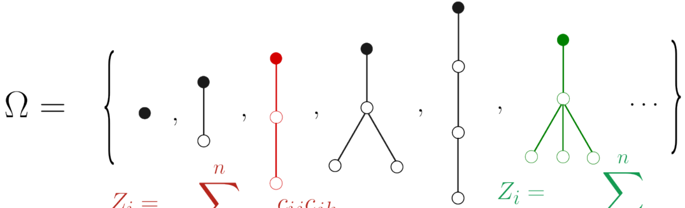
The Fourier tree basis



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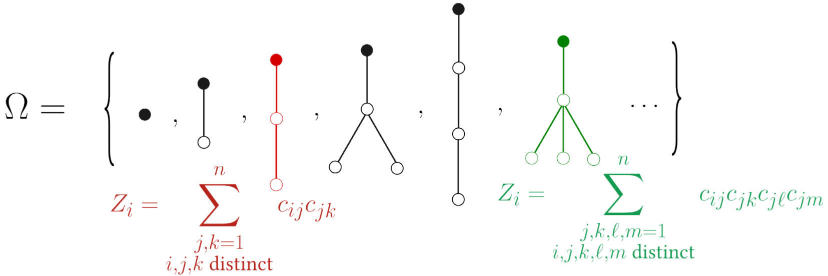

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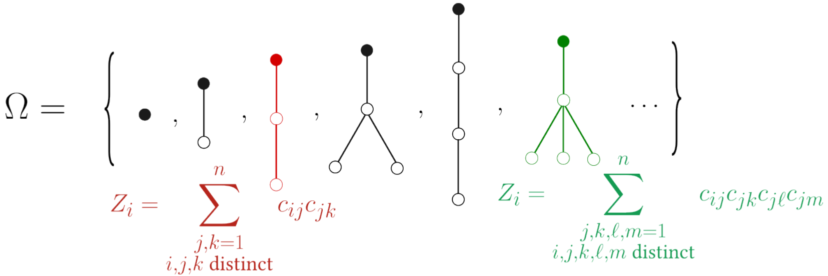
[state evolution]

The Fourier tree basis



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✓ Simple analysis of algorithms for maximizing random polynomials [state evolution]

✓ New expression for the value that they achieve [extended Parisi formula]

Conclusion

1. Beyond trace method & flattenings $\implies$ spectral theory for hypergraphs.
2. Maximizing non-homogeneous polynomials?
3. Full power of the Fourier tree basis?

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Thank you, Luca.