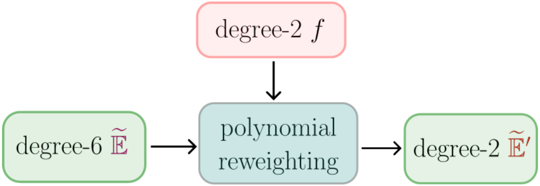
# New SDP Roundings and Certifiable Approximation for Cubic Optimization

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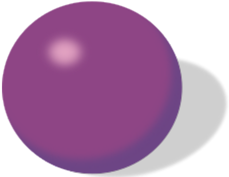
# Quadratic polynomial optimization

**Question:** Given an arbitrary homogeneous degree-2 multilinear polynomial

$$p(\mathbf{x}) := p(x_1, \ldots, x_n) = \sum_{1 \leqslant i < j \leqslant n} A_{ij} x_i x_j \,,$$

can we approximate efficiently $\max p(\mathbf{x})$?

$$\max_{\|\mathbf{x}\|_2 = 1} p(\mathbf{x})$$



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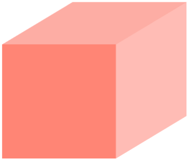

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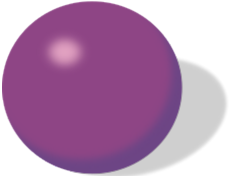

Largest eigenvalue of

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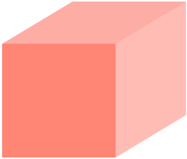

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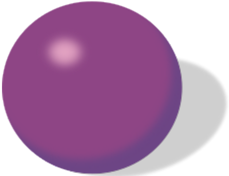

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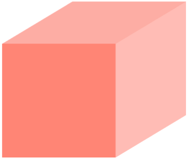

[Charikar-Wirth'04] $O(\log n)$-approximation

[Grothendieck'53, ..., Alon-Naor'04]
$O(1)$-approximation when $p(\mathbf{x}, \mathbf{y}) = \sum_{ij} A_{ij} x_i y_j$

Based on rounding the basic SDP relaxation

# Our main results

**Question:** Given an arbitrary homogeneous degree-3 multilinear polynomial

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Over both $S^{n-1}$ and $\{-1, 1\}^n$:

**Thm 1:** [HKPT'24] $O(\sqrt{n})$-approximation in time $n^{O(1)}$ via a "canonical" SDP relaxation

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$\longrightarrow$ generalizes [Bhattiprolu-Gosh-Guruswami-Lee-Tulsiani'17]

**Thm 2:** [HKPT'24] $O(\sqrt{n/\mathbf{k}})$-approximation in time $2^{O(\mathbf{k})} n^{O(1)}$ via a "pruned" SDP relaxation

$\longrightarrow$ matches [Khot-Naor'07] for $\mathbf{k} = \log n$

$\longrightarrow$ also provides a *certifiable upper bound* on $\max p(\mathbf{x})$ (the SDP dual)

# Decoupling interlude

Why *cubic* optimization?

[Khot-Naor'07] Let $\mathbf{A} = (A_{ijk})$ be a symmetric 3-tensor with zero "diagonal",

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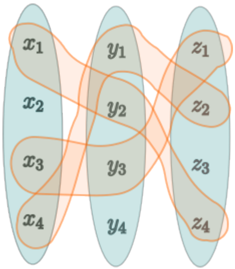


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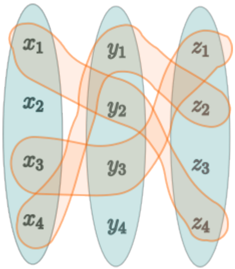

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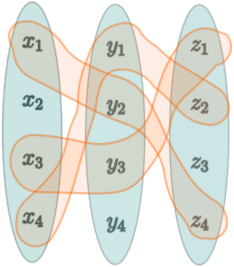

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Only holds for *odd-degree* (multilinear) polynomials

# Result 1: the canonical sum-of-squares relaxation

**Thm 1:** Let $p(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{ijk} A_{ijk} x_i y_j z_k$. Rounding the degree-6 sum-of-squares SDP relaxation yields a $O(\sqrt{n})$-approximation to $\max_{\mathbf{x},\mathbf{y},\mathbf{z} \in \{-1,1\}^n} p(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

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**The SDP relaxation:**

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$$\widetilde{\mathbb{E}} : \{\text{polynomials of degree} \leqslant 6 \text{ in } \mathbf{x}, \mathbf{y}, \mathbf{z}\} \to \mathbb{R}$$

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**Ex:** if $f$ is a polynomial in $\mathbf{x},\mathbf{y},\mathbf{z}$ of degree $\leqslant 3$, $|\widetilde{\mathbb{E}} f| \leqslant (\widetilde{\mathbb{E}} f^2)^{\frac{1}{2}}$

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**Input:** degree-6 $\widetilde{\mathbb{E}}$ maximizing $\widetilde{\mathbb{E}}\,[p(\mathbf{x}, \mathbf{y}, \mathbf{z})]$

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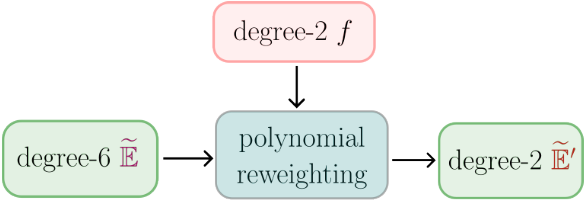


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1. Draw at random $\bar{\mathbf{x}} \sim \{-1,1\}^n$
2. **Reweight $\widetilde{\mathbb{E}}$ into $\widetilde{\mathbb{E}}'$ based on $f = p(\bar{\mathbf{x}},\mathbf{y},\mathbf{z})$**
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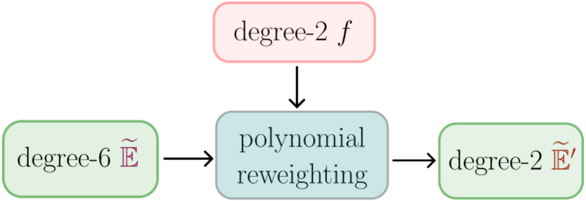


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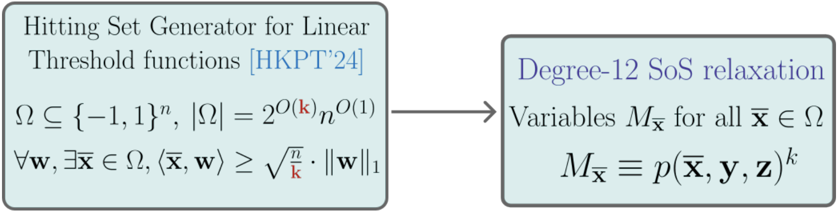

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The advantage over $\frac{7}{8}$ is a *non-homogeneous* polynomial.

- Optimize separately the degree-1, 2, 3 parts.
- Add specific constraints to the SDP to avoid cancellations.

# Conclusion

**Problem:** maximize a homogeneous degree-3 polynomial over $S^{n-1}$ or $\{\pm 1\}^n$

- Rounding of the canonical SoS hierarchy via *polynomial reweightings*.
- Slightly improved time/approximation tradeoff via a *pruned SDP relaxation*.
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**Open problems:**

- Best known integrality gap for cubic optimization: $n^{1/4}$
- $\sqrt{n}$-approximation for *non-satisfiable* 3SAT instances?
- Applications of our rounding techniques to other problems?