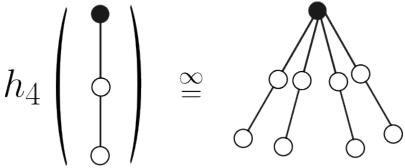
# Understanding iterative algorithms with Fourier diagrams




*Lucas Pesenti*

Bocconi University

Chris Jones

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I am currently looking for a **postdoc** (starting Sep. 2025/Jan. 2026)

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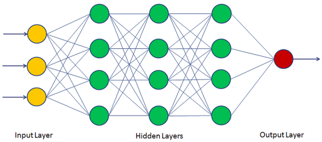


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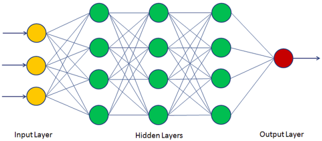


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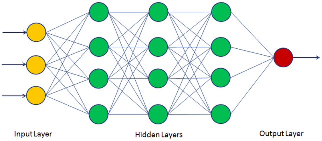


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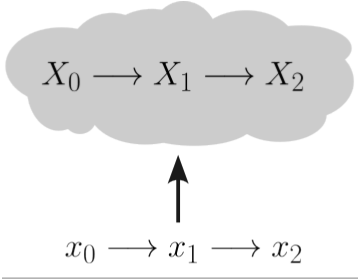
*Today*: idealized iteration $\mathbf{X_0}, \mathbf{X_1}, \ldots$

**The tree approximation**

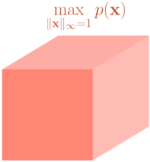


# Plan

1. Motivation: random polynomial optimization
2. Building the tree approximation
3. Working in the asymptotic tree basis

# Motivation: random polynomial optimzation

# Quadratic polynomial optimization




**Problem:** maximize

$$p(\mathbf{x}) = \sum_{i,j=1}^{n} A_{ij} x_i x_j \,.$$

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# Quadratic polynomial optimization

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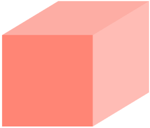

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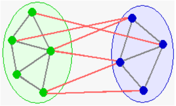

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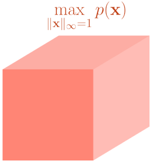


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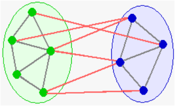

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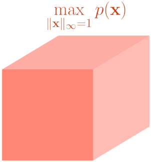


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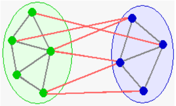

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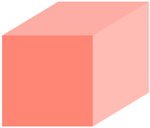

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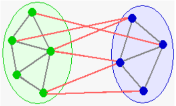

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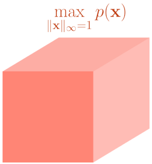


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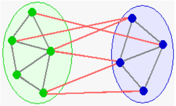

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[Montanari'21] polytime algorithm achieving w.h.p. $(1-\epsilon)$-approximation for any fixed $\epsilon > 0$

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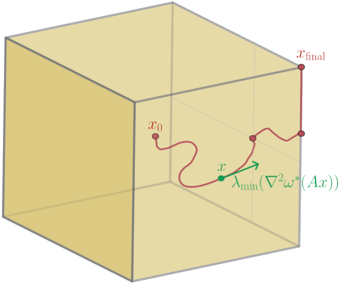

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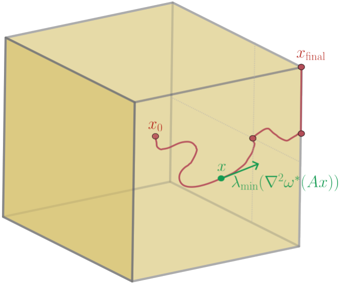


Iterative algorithms for *non-certifiable* optimization problems

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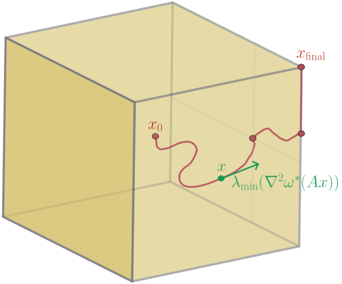


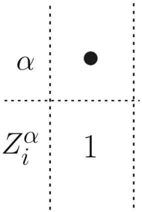

Iterative algorithms for *non-certifiable* optimization problems

[**P**-Vladu'23] discrepancy theory

# Building the tree approximation

# The Fourier diagram basis

The Fourier diagram basis $\{\mathbf{Z}^\alpha \in \mathbb{R}^n : \alpha \text{ unlabeled rooted graph}\}$



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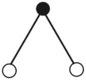
| $\alpha$ |  |  |  |
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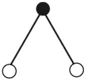
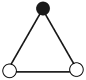
| $\alpha$ |  |  |  |  |
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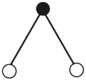
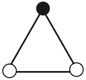
| $\alpha$ |  |  |  |  |
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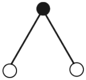
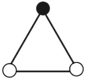
| $\alpha$ |  |  |  |  |
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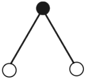
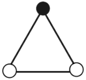
| $\alpha$ |  |  |  |  |
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- Lower bounds against low degree polynomials & SDP hierarchies.
- Important to sum over distinct indices

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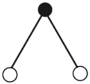

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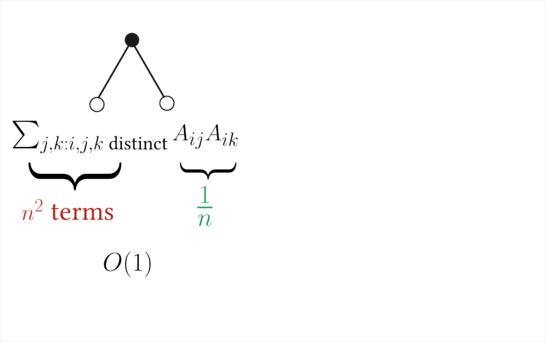

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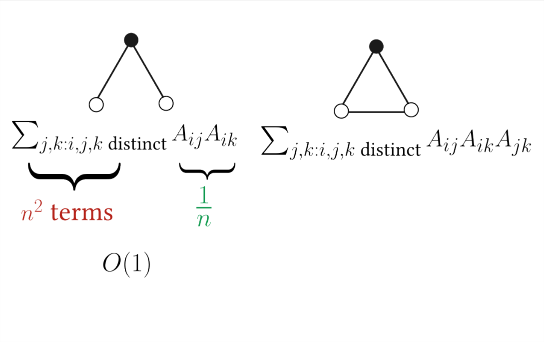

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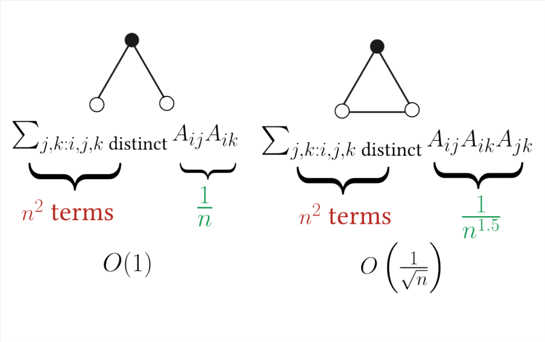

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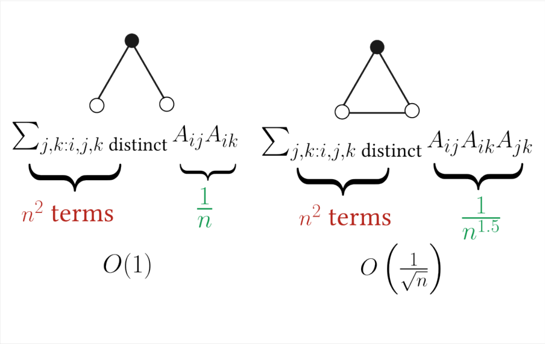


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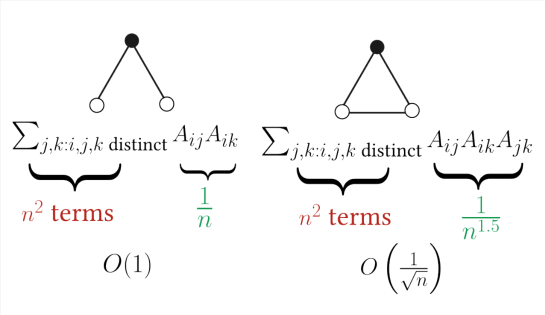


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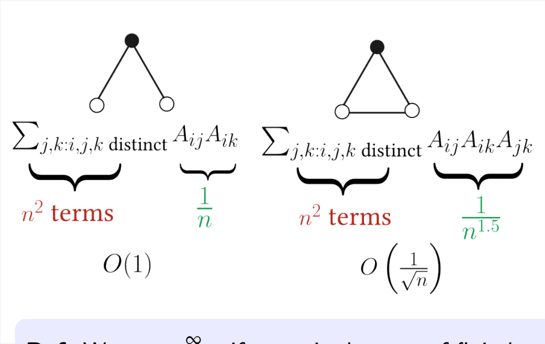
**Def:** We say $\mathbf{x} \overset{\infty}{=} \mathbf{y}$ if $\mathbf{x} - \mathbf{y}$ is the sum of finitely many cyclic diagrams.

# Cyclic diagrams

If $A_{ij} \underset{\text{i.i.d.}}{\sim} \pm\frac{1}{\sqrt{n}}$, the Fourier diagram basis simplifies as $n \to \infty$.

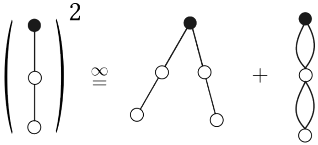



**Thm:** [Jones-**P**'24]
The cyclic diagrams are negligible as $n \to \infty$.

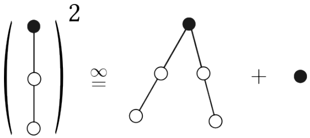
**Def:** We say $\mathbf{x} \overset{\infty}{=} \mathbf{y}$ if $\mathbf{x} - \mathbf{y}$ is the sum of finitely many cyclic diagrams.

- Only for diagrams of size $O(1)$
- In general: free cumulants

# The asymptotic tree basis



# The asymptotic tree basis



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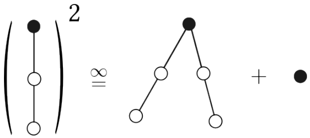


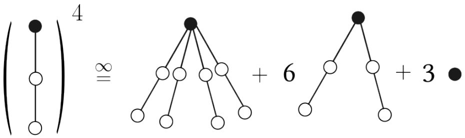

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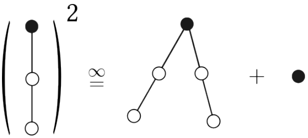
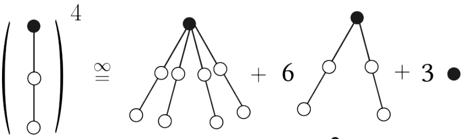
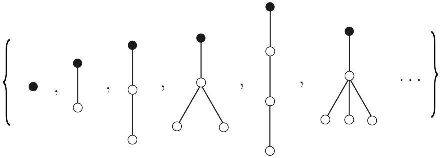




**Thm:** [Jones-**P** '24]
The tree diagrams with **one subtree** at the root are asymptotically independent Gaussian vectors



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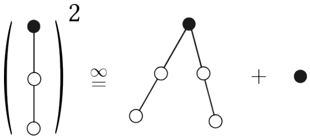

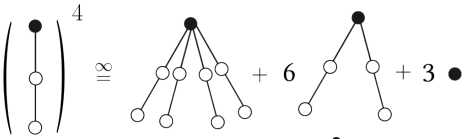

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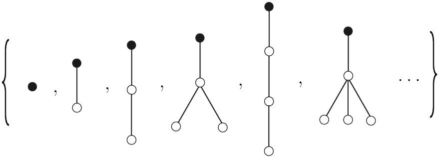

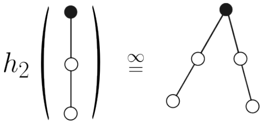

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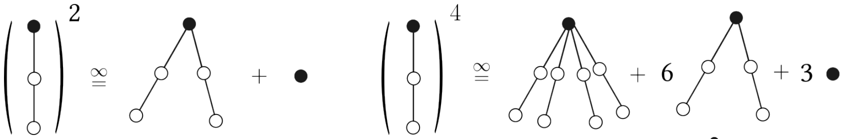

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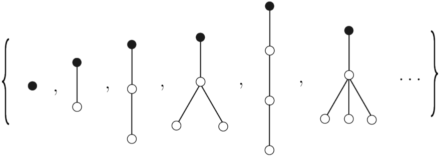

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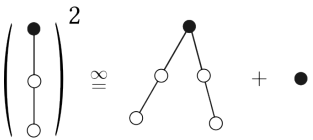

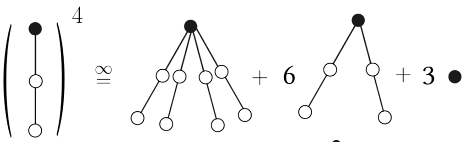

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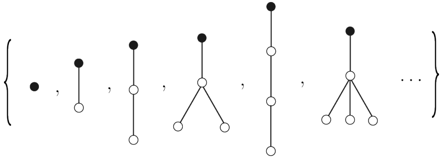
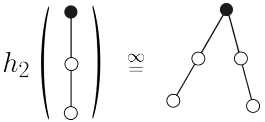
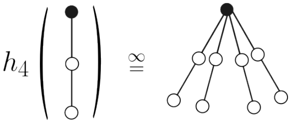
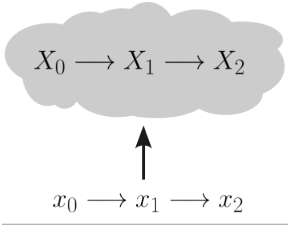






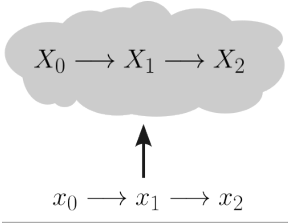
**Thm:** [Jones-**P** '24]
The tree diagrams with **several subtrees** at the root are asymptotically Hermite polynomials in the Gaussians

# The tree approximation




$\mathbf{x_t} = \mathbf{A}\mathbf{x_{t-1}}$ or $\mathbf{x_t} = f_t(\mathbf{x_{t-1}})$

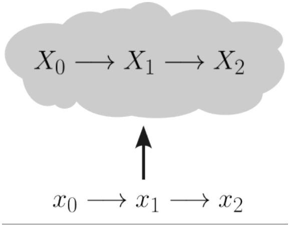
# The tree approximation




$$\mathbf{x_t} = \mathbf{A}\mathbf{x_{t-1}} \quad \text{or} \quad \mathbf{x_t} = f_t(\mathbf{x_{t-1}})$$

1. Expand $\mathbf{x_t}$ in the Fourier diagram basis

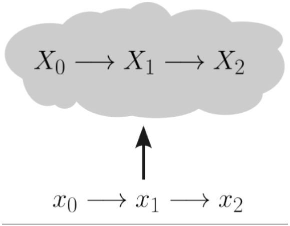
# The tree approximation




$$\mathbf{x_t} = \mathbf{A}\mathbf{x_{t-1}} \quad \text{or} \quad \mathbf{x_t} = f_t(\mathbf{x_{t-1}})$$

1. Expand $\mathbf{x_t}$ in the Fourier diagram basis
2. $\mathbf{X_t}$ := expansion restricted to tree diagrams.

# The tree approximation

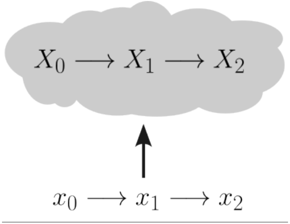



$$\mathbf{x_t} = \mathbf{A}\mathbf{x_{t-1}} \quad \text{or} \quad \mathbf{x_t} = f_t(\mathbf{x_{t-1}})$$

1. Expand $\mathbf{x_t}$ in the Fourier diagram basis
2. $\mathbf{X_t} :=$ expansion restricted to tree diagrams.

Then: $\mathbf{x_t} \overset{\infty}{=} \mathbf{X_t}$, so $\|\mathbf{x_t} - \mathbf{X_t}\|_\infty = O\left(\frac{1}{\sqrt{n}}\right)$.

# The tree approximation

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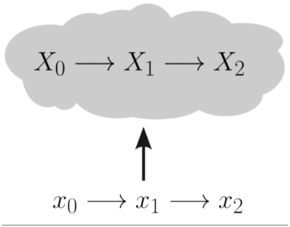
$\mathbf{X_t}$ follows a *simplified Gaussian dynamic*!

# The tree approximation

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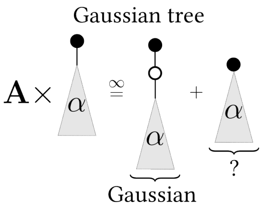

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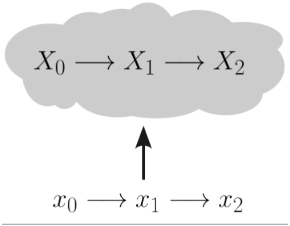


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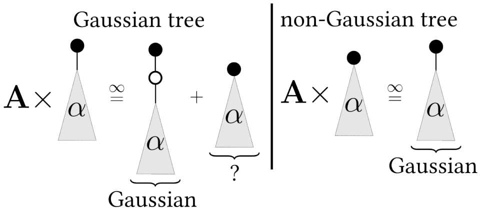

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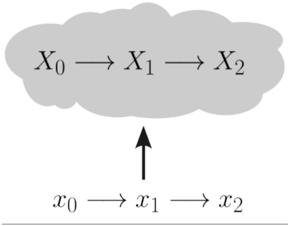


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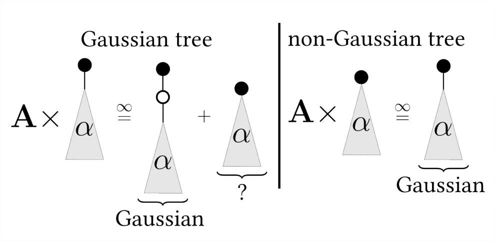


**Lem:** [Jones-**P** '24]

$$\mathbf{A}\mathbf{X} \overset{\infty}{=} \mathbf{X}^+ + \mathbf{X}^-$$

## Working in the asymptotic tree basis

# The cavity method

## Belief propagation:

$$m_{i\to j}^{t+1} = f_t\left(\sum_{k\neq i} A_{ik}\, m_{k\to i}^{t}\right)\,, \quad m_i^{t+1} = g_t\left(\sum_{k=1}^{n} A_{ik}\, m_{k\to i}^{t}\right)\,.$$

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Belief propagation:

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Approximate message passing:

$$\mathbf{w}^{\mathbf{t+1}} = \mathbf{A} f_t\left(\mathbf{w}^{\mathbf{t}}\right) - \frac{1}{n}\sum_{i=1}^{n} {f_t}'(w_i^t)\mathbf{w}^{\mathbf{t-1}}, \quad \mathbf{m}^{\mathbf{t}} = g_t(\mathbf{w}^{\mathbf{t}}).$$

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$$\mathbf{w}^{\mathbf{t+1}} = \mathbf{A} f_t\left(\mathbf{w}^{\mathbf{t}}\right) - \frac{1}{n}\sum_{i=1}^{n} f_t{}'(w_i^t)\mathbf{w}^{\mathbf{t-1}}\,, \quad \mathbf{m}^{\mathbf{t}} = g_t(\mathbf{w}^{\mathbf{t}})\,.$$

**Thm:** [Bayati-Lelarge-Montanari '11, Jones-**P.** '24] $\mathbf{m}^{\mathbf{t},\mathrm{BP}} \overset{\infty}{=} \mathbf{m}^{\mathbf{t},\mathrm{AMP}}$.

# The cavity method

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$$m_{i \to j}^{t+1} = f_t \left( \sum_{k \neq i} A_{ik} m_{k \to i}^t \right), \quad m_i^{t+1} = g_t \left( \sum_{k=1}^{n} A_{ik} m_{k \to i}^t \right).$$

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$$\mathbf{w}^{\mathbf{t}+1} = \mathbf{A} f_t \left( \mathbf{w}^{\mathbf{t}} \right) - \frac{1}{n} \sum_{i=1}^{n} f_t{}'(w_i^t) \mathbf{w}^{\mathbf{t-1}}, \quad \mathbf{m}^{\mathbf{t}} = g_t(\mathbf{w}^{\mathbf{t}}).$$

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$$m_{i \to j}^{t+1} = f_{t+1} \left( w_i^{t+1} - A_{ij} m_{j \to i}^t, \ldots, w_{k \to i}^1 - A_{ij} m_{j \to i}^0, w_i^0 \right).$$

Given that the entries $A_{ij}$ are on the scale of $1/\sqrt{n}$, which we expect to be much smaller than the magnitude of the messages, we perform a first-order Taylor approximation (the partial derivatives are with respect to the coordinates of $f_{t+1}$ and the last coordinate is ignored because $w_i^0$ is constant):

$$m_{i \to j}^{t+1} \approx f_{t+1} \left( w_i^{t+1}, \ldots, w_i^1, w_i^0 \right) - A_{ij} \sum_{s=1}^{t+1} m_{j \to i}^{s-1} \frac{\partial f_{t+1}}{\partial w^s} \left( w_i^{t+1}, \ldots, w_i^1, w_i^0 \right). \qquad (*)$$

Plugging this approximation in the definition of $w_i^{t+1}$,

$$\begin{aligned} w_i^{t+1} &\approx \sum_{k=1}^{n} A_{ik} f_t(w_k^t, \ldots, w_k^0) - \sum_{k=1}^{n} A_{ik}^2 \sum_{s=1}^{t} m_{i \to k}^{s-1} \frac{\partial f_t}{\partial w^s}(w_k^t, \ldots, w_k^0) \\ &\approx \sum_{k=1}^{n} A_{ik} f_t(w_k^t, \ldots, w_k^0) - \sum_{k=1}^{n} \frac{1}{n} \sum_{s=1}^{t} f_{s-1}(w_i^{s-1}, \ldots, w_i^0) \frac{\partial f_t}{\partial w^s}(w_k^t, \ldots, w_k^0) \qquad (**) \\ &= \sum_{k=1}^{n} A_{ik} f_t(w_k^t, \ldots, w_k^0) - \sum_{s=1}^{t} b_{s,t} f_{s-1}(w_i^{s-1}, \ldots, w_i^0). \end{aligned}$$

This shows that $w_i^{t+1}$ approximately satisfies the AMP recursion Eq. (11), as desired.

# The cavity method

## Belief propagation:

$$m_{i\to j}^{t+1} = f_t\left(\sum_{k\neq i} A_{ik} m_{k\to i}^t\right), \quad m_i^{t+1} = g_t\left(\sum_{k=1}^n A_{ik} m_{k\to i}^t\right).$$

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This shows that $w_i^{t+1}$ approximately satisfies the AMP recursion Eq. (11), as desired.

**Pf**: replace $\approx$ by $\overset{\infty}{=}$!

# The cavity method

Belief propagation:

$$m_{i\to j}^{t+1} = f_t\left(\sum_{k\neq i} A_{ik} m_{k\to i}^t\right), \quad m_i^{t+1} = g_t\left(\sum_{k=1}^{n} A_{ik} m_{k\to i}^t\right).$$

Approximate message passing:

$$\mathbf{w}^{\mathbf{t}+\mathbf{1}} = \mathbf{A} f_t\left(\mathbf{w}^{\mathbf{t}}\right) - \frac{1}{n}\sum_{i=1}^{n} f_t{}'(w_i^t)\mathbf{w}^{\mathbf{t}-\mathbf{1}}, \quad \mathbf{m}^{\mathbf{t}} = g_t(\mathbf{w}^{\mathbf{t}}).$$

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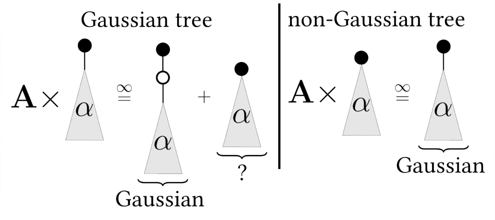
This shows that $w_i^{t+1}$ approximately satisfies the AMP recursion Eq. (11), as desired.

**Pf**: replace $\approx$ by $\overset{\infty}{=}$!

**Lem:** incoming messages $(m_{i\to j}^t)_{i:i\neq j}$ are asymptotically independent

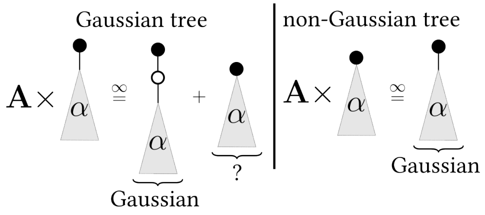
"Cavity method" made rigorous

## State evolution




$$\mathbf{A}\mathbf{X} \overset{\infty}{=} \mathbf{X}^{+} + \mathbf{X}^{-}$$

# State evolution




$$\mathbf{A}\mathbf{X} \overset{\infty}{=} \mathbf{X}^{+} + \mathbf{X}^{-}$$

Approximate message passing:

$$\mathbf{X_{t+1}} = f_t(\mathbf{X_t}, \ldots, \mathbf{X_0})^{+}$$

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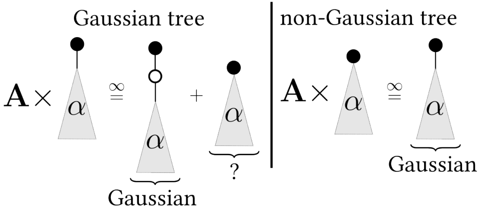


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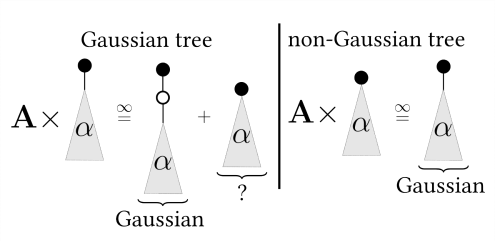


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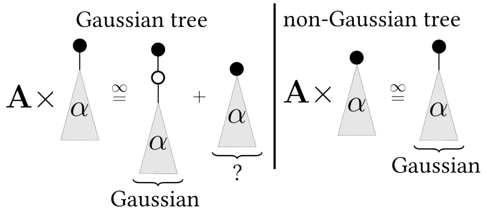


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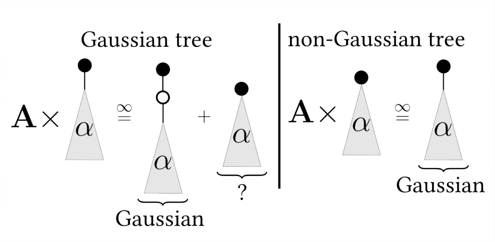


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**Thm:** (*state evolution*) [Bolthausen, Javanmard-Montanari, ...]

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**Problem:** given $A_{ij} \underset{\text{i.i.d.}}{\sim} \pm\frac{1}{\sqrt{n}}$, maximize $\langle \mathbf{x}, \mathbf{Ax} \rangle$ over $\mathbf{x} \in \{\pm 1\}^n$

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[Montanari '21] for some well-chosen $f_t$,

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$$\langle \mathbf{X}, \mathbf{A}\mathbf{X} \rangle \overset{\infty}{=} \langle \mathbf{X}, \mathbf{X}^+ \rangle + \langle \mathbf{X}, \mathbf{X}^- \rangle \overset{\infty}{=} 2\langle \mathbf{X}, \mathbf{X}^+ \rangle$$

**Question:** "combinatorial" Parisi dual representation?

$$\lim_{\delta \to 0} \sup_{X : \mathbb{P}(X \in [-1,1]) \geqslant 1-\delta} 2\mathbb{E} X X^+ = \text{Parisi constant}\,?$$

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The Fourier diagram basis simplifies as $n \to \infty$ to the asymptotic tree basis of independent Gaussian vectors.

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**Theorem 6.2.** *Suppose that $A = A(n)$ satisfies Assumption 6.1 and generate $x_t$ according to Eq. (16). Then there exist universal constants $c, \delta > 0$ such that for all $t \le cn^\delta$,*

$$\|x_t - Z_{t\text{-path}}\|_\infty \xrightarrow{a.s.} 0\,.$$

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**Thank you!**