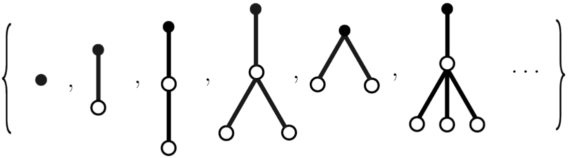
Algorithms beyond the union bound

Polynomial optimization and discrepancy theory

$\Omega = \mathrm{span}$


Lucas Pesenti

January 19, 2026

Advisor: Laura Sanità
Co-advisor: Pravesh Kothari

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Polynomial optimization and discrepancy theory

Ex: Boolean formula

$$
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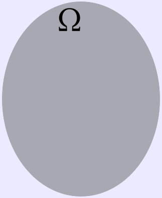

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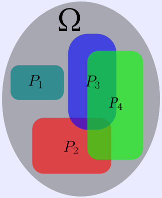

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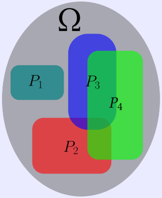

Union bound:

$$\sum_i \Pr(P_i) < 1$$

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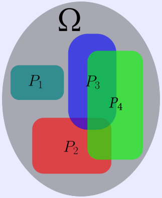

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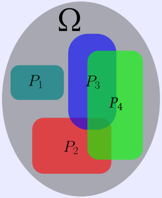

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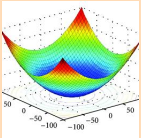
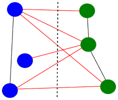
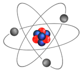


This thesis:
Optimization approach

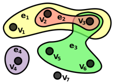
Plan



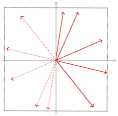
1. Constraint satisfaction problems



2. Random polynomials



3. Spectral hypergraph theory



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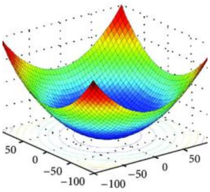

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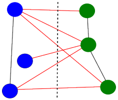

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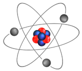

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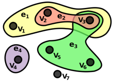

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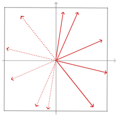

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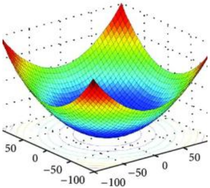

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State-of-the-art: 

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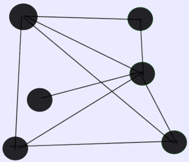
Large cuts in graphs:



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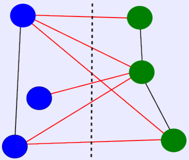

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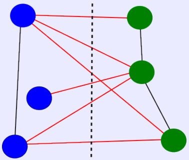

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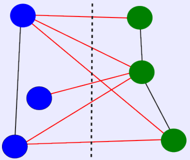

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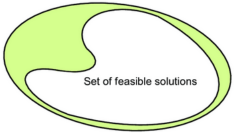

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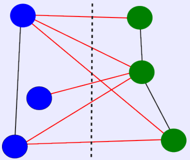

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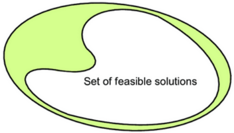

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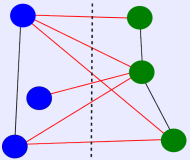

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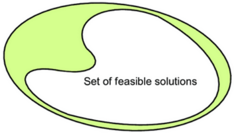

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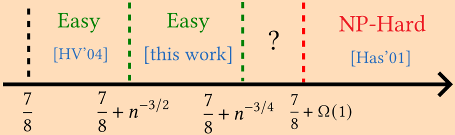
This thesis: improve over 



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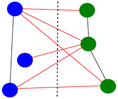

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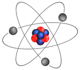

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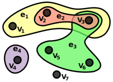

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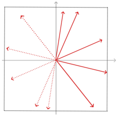

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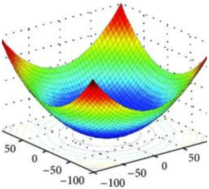
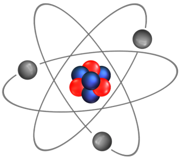


2. Random polynomials (1/2)



Typical/random coefficients $c_{ij} = c_{ji} \sim \mathcal{N}(0,1)$

$$\mathrm{OPT}(n) := \max_{\|\boldsymbol{x}\|_2 \leqslant 1} \sum_{i,j=1}^{n} c_{ij} x_i x_j \qquad\qquad \mathrm{OPT}(n) := \max_{\|\boldsymbol{x}\|_\infty \leqslant 1} \sum_{i,j=1}^{n} c_{ij} x_i x_j$$

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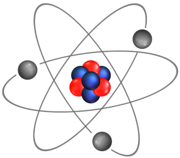

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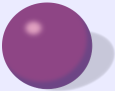

Union bound:
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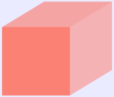

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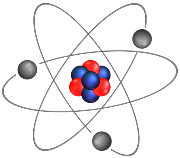

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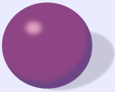

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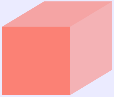

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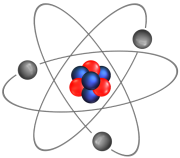

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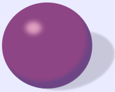

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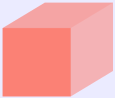

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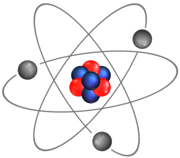

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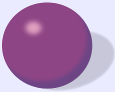

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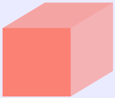

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Rigorous methods:

- very technical
- hard to generalize

2. Random polynomials (2/2)

This thesis: as $n \to \infty$,

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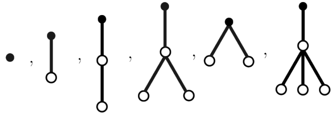
$\Omega = \operatorname{span}\Big\{$  $\cdots \Big\}$

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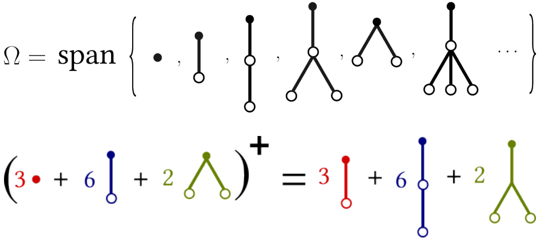
$$\geqslant$$

$$\sup_{\substack{X \in \Omega \\ \|X\|_p \leqslant 1}} \langle X, X^+ \rangle$$



2. Random polynomials (2/2)

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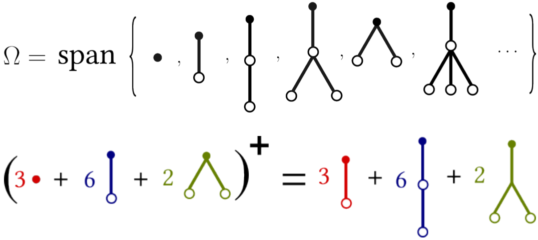
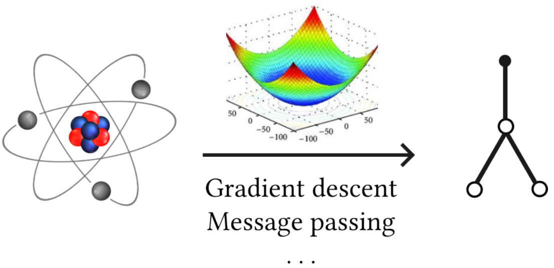
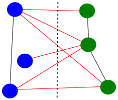
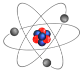




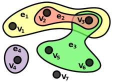
Plan



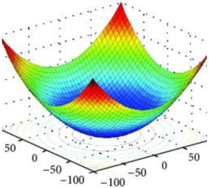
1. Constraint satisfaction problems

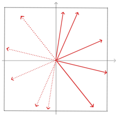


2. Random polynomials



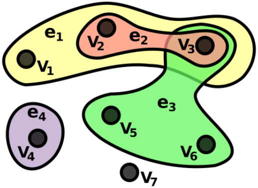
3. Spectral hypergraph theory





4. Combinatorial discrepancy

3. Spectral hypergraph theory



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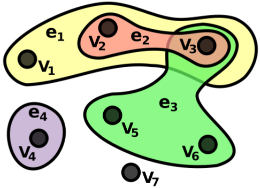

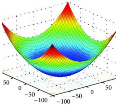

[Friedman–Wigderson'94]

Def: H has a spectral gap if

$$\underbrace{\max_{\|\boldsymbol{x}\|_2 \leqslant 1} p_H(\boldsymbol{x})}_{\lambda_2(H)} \ll \underbrace{\max_{\|\boldsymbol{x}\|_2 \leqslant 1} q_H(\boldsymbol{x})}_{\lambda_1(H)}$$

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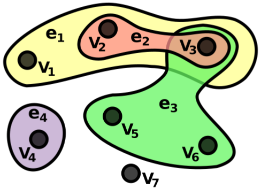

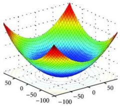

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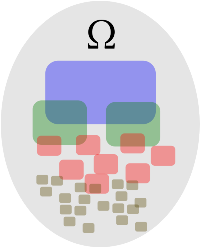
✗ Naive union bound



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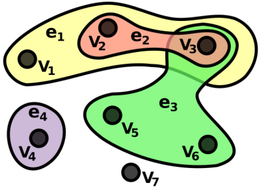

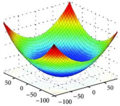

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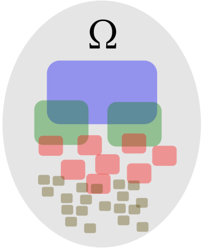
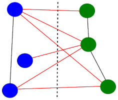


This thesis: Sparse random hypergraphs have a spectral gap once $\lambda_1(H) \to \infty$

Plan



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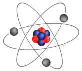

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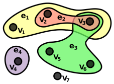

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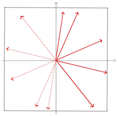
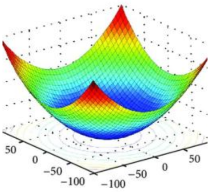


4. **Combinatorial discrepancy**



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$$\textbf{Minimize:}\quad \operatorname{disc}(x_1, \ldots, x_n) := \left\| \sum_{i=1}^{n} x_i \boldsymbol{u}_i \right\|$$

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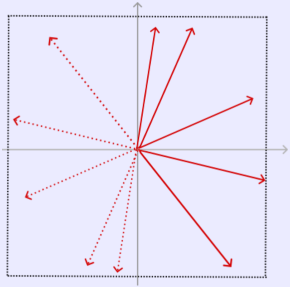
$$\min_{\boldsymbol{x} \in \{-1,1\}^n} \left\| \sum_{i=1}^{n} x_i \boldsymbol{u}_i \right\|_\infty \leqslant K\sqrt{n}$$



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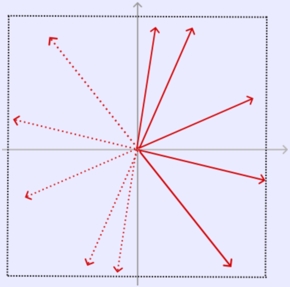

Union bound: $K \leqslant \sqrt{\log n}$

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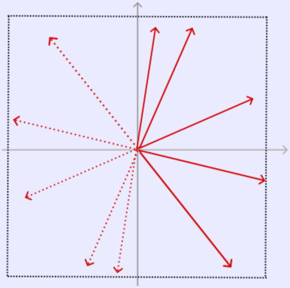

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[Spencer'85] $K \leqslant 6$

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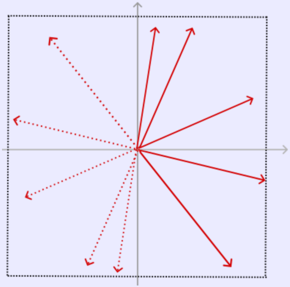

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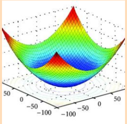

This thesis:
$K \leqslant 4.1$

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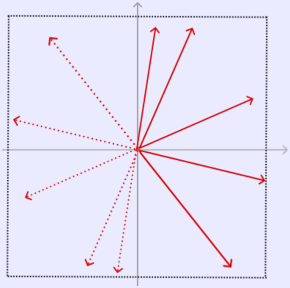

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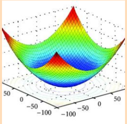

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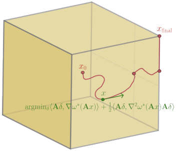
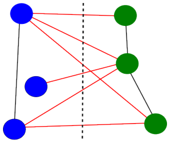


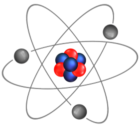
Newton's method on a regularized objective

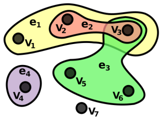
Algorithms beyond the union bound

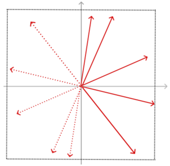
Polynomial optimization and discrepancy theory

"To prove that something exists, analyze an algorithm finding it"









Algorithms beyond the union bound

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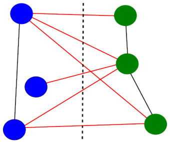

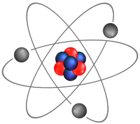

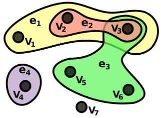

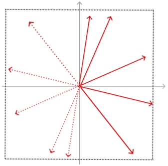

Future work:

1. Unify analysis of higher-degree convex relaxations
2. Solve optimization problems in the tree basis
3. Generalizations to semi-random and deterministic polynomials

Algorithms beyond the union bound

Polynomial optimization and discrepancy theory

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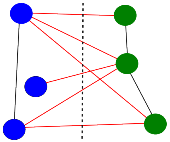

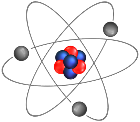

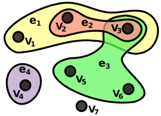

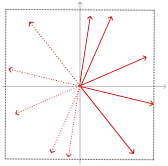

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