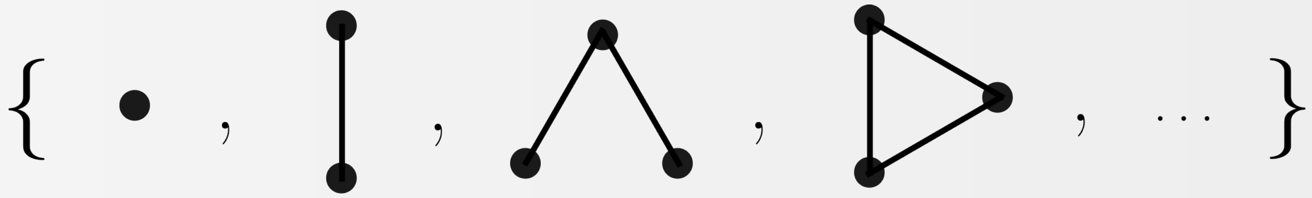
Diagram analysis of iterative algorithms

Chris Jones   Lucas Pesenti

The diagram basis

The diagram basis $\{Z_\alpha : \alpha \text{ graph}\}$ associated to an $n \times n$ matrix $\mathbf{A}$ is:

$$Z_\alpha(\mathbf{A}) := \sum_{\substack{\phi: V(\alpha) \to [n] \\ \phi \text{ injective}}} \prod_{\{u,v\} \in E(\alpha)} A_{\phi(u),\phi(v)} .$$



1 $\quad \sum_{i \neq j} A_{ij}$ $\quad \sum_{i \neq j \neq k} A_{ij} A_{jk}$ $\quad \sum_{i \neq j \neq k} A_{ij} A_{jk} A_{ki}$

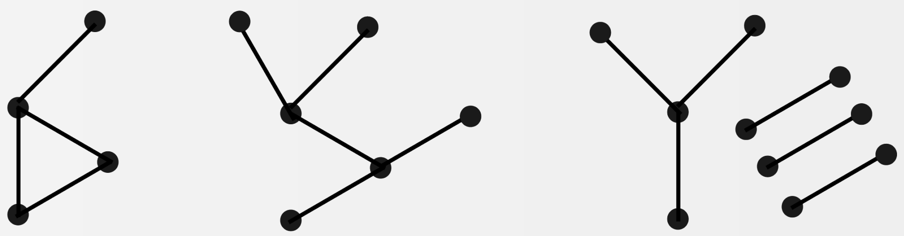
$\{Z_\alpha : |E(\alpha)| \leq d\}$ spans all symmetric degree-d polynomials in the entries of $\mathbf{A}$.

Instrumental in proving lower bounds against low degree polynomials & SDP hierarchies.

The tree approximation

Main theorem: if $\mathbf{A}$ has independent mean-0 variance-1 entries, as $n \to \infty$,

- The cyclic diagrams are negligible.
- The trees are Gaussians.
- The forests are Hermite polynomials in these Gaussians.



The cavity method

Using a vector variant of the diagram basis, we can make some heuristic arguments directly rigorous!

$$[\text{BP}] \quad m^t_{i\to j} = f_t\left(\sum_{k\neq i} A_{ik} m^{t-1}_{k\to i}\right), \qquad m^t_i = g_t\left(\sum_{k=1}^{n} A_{ik} m^{t-1}_{k\to i}\right).$$

$$[\text{AMP}] \quad \mathbf{w}^t = \mathbf{A} f_t\left(\mathbf{w}^{t-1}\right) - \text{Onsager term}, \qquad \mathbf{m}^t = g_t(\mathbf{w}^t).$$

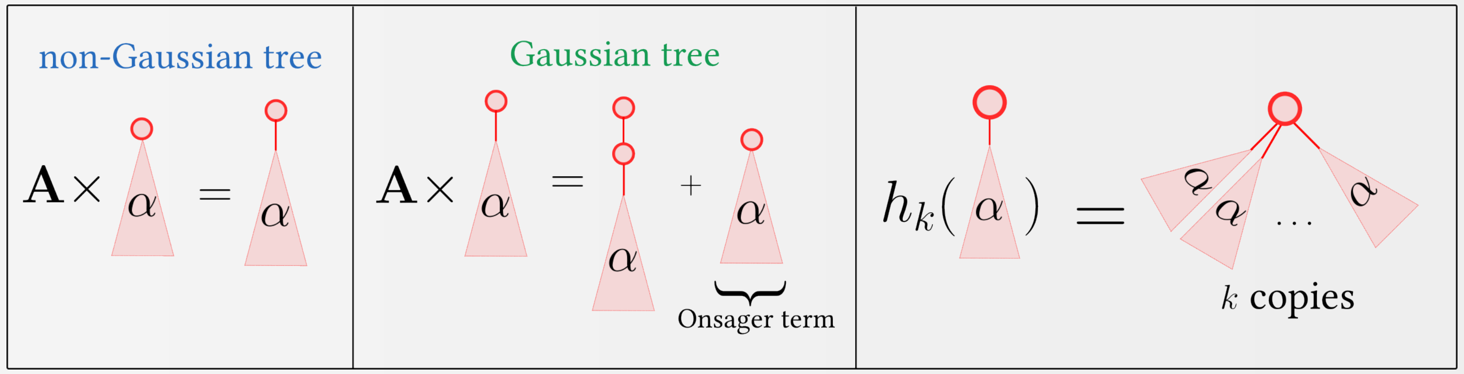
Expand the messages in the diagram basis.

$\mathbf{m}^{t,\mathrm{BP}} - \mathbf{m}^{t,\mathrm{AMP}}$ is a sum of cyclic diagrams.

Thus BP & AMP are asymptotically equivalent.

State evolution

Effects of the BP/AMP operations on rooted trees:



Easy pictorial proof of state evolution!

Open questions

1. What is the right diagram basis for rotationally invariant distributions?
2. How to handle a number of iterations growing with n?
3. How much randomness is needed for the tree approximation?